

Upper and Lower Bounds on Gravitational Entropy

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Abstract

The gravitational entropy of the Universe is large and subtle to calculate. A lower bound is well known from supermassive black holes at the centers of galaxies, but the remainder is harder to pin down. A parametric model of clumped matter entropy is provided. We suggest a new upper bound due to dark matter halos, that is far below the holographic bound, yet above that for the supermassive black holes.

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Introduction.

As interest grows in pursuing alternatives to the Big Bang, including cyclic cosmologies, it becomes more pertinent to address the difficult question of what is the present entropy of the universe?

Entropy is particularly relevant to cyclicity because it does not naturally cycle but has the propensity to increase monotonically because of the second law of thermodynamics. In one recent proposal the entropy is jettisoned at turnaround. In any case, for cyclicity to be possible there must be a gigantic reduction in entropy, at deflation in [1], of the visible universe at some time during each cycle.

Standard treatises on cosmology [2,3] address the question of the entropy of the universe and arrive at a generic formula for a thermalized gas of the form

$$S = \frac{2\pi^2}{45} g_* V_U T^3 \quad (1)$$

where g_* is the number of degrees of freedom, T is the Kelvin temperature and V_U is the volume of the visible universe. From Eq.(1) with $T_\gamma = 2.7$ K and $T_\nu = T_\gamma(4/11)^{1/3} = 1.9$ K we find the entropy in CMB photons and neutrinos are roughly equal today

$$S_\gamma(t_0) \sim S_\nu(t_0) \sim 10^{88}. \quad (2)$$

Our topic here is the gravitational entropy, $S_{grav}(t_0)$. Following the same path as in Eqs.(1,2) we obtain for a thermal gas of gravitons $T_{grav} = 0.9$ K and then

$$S_{grav}^{(thermal)}(t_0) \sim 10^{86} \quad (3)$$

which is a couple of orders of magnitude below that for photons and neutrinos. On the other hand, while radiation thermalizes at $T \sim 0.1$ eV for which the measurement of the black body spectrum provides good evidence and there is every reason, though no direct evidence, to expect that the relic neutrinos were thermalized at $T \sim 1$ MeV, the thermal equilibration of the present gravitons is less definite. If gravitons did thermalize, it was at or above the Planck scale, $T \gtrsim 10^{19}$ GeV, when everything is uncertain because of quantum gravity effects. If the gravitons are in a non-thermalized gas their entropy will be lower than in Eq.(3), for the same number density. But there are larger contributions to gravitational entropy from elsewhere.

In this paper we will review upper and lower bounds on the gravitational entropy of the Universe and then argue that dark matter halos contain the dominant component. While a detailed model based on halo distributions is not described, we do provide a parametrization of the halo entropy and give a one parameter bound that is sufficient for our purposes. The model is a simplification in that it assumes all halos have equal masses and ignores cluster and supercluster halos. Hence our results will be semi-quantitative.

Relaxing these approximations would unduly complicate the model without significantly improving any insight it hopefully provides.

Upper Limit on Gravitational Entropy.

We shall assume that dark energy has zero entropy and therefore we must concentrate on the gravitational entropy associated with dark matter. ^{#3} The dark matter is clumped into halos with typical mass $M(halo) \simeq 10^{11}M_\odot$ and radius $R(halo) = 10^4 pc \simeq 3 \times 10^{17} km \simeq 10^{17} r_S(M_\odot)$, where $M_\odot \simeq 10^{57} GeV \simeq 10^{30} kg$ is the solar mass and $r_S(M_\odot)$ its Schwarzschild radius. There are, say, 10^{12} halos in the visible universe whose total mass is $\simeq 10^{23}M_\odot$ and corresponding Schwarzschild radius is $r_S(10^{23}M_\odot) \simeq 3 \times 10^{23} km \simeq 10$ Gpc. This coincides with the radius of the visible universe corresponding to the critical density and has led to an upper limit for the gravitational entropy of one black hole with mass $M_U = 10^{23}M_\odot$. Using $S_{BH}(\eta M_\odot) \simeq 10^{77} \eta^2$ corresponds to the holographic principle [5, 6] for the upper limit on the gravitational entropy of the visible universe:

$$S_{grav}(t_0) \lesssim S_{grav}^{(HOLO)}(t_0) \simeq 10^{123} \quad (4)$$

which is 37 orders of magnitude greater than for the thermalized graviton gas in Eq.(3) and leads us to suspect (correctly) that Eq.(3) is a gross underestimate. Nevertheless, Eq.(4) does provide a credible upper limit, an overestimate yet to be refined downwards below, on the quantity of interest, $S_{grav}(t_0)$.

The reason why a thermalized gas of gravitons grossly underestimates the gravitational entropy is because of the 'clumping' effect of gravity on entropy [7]. Because gravity is universally attractive its entropy is increased by clumping. This is somewhat counter-intuitive since the opposite is true for the more familiar ideal gas. It is best illustrated by the fact that a black hole always has maximal entropy by virtue of the holographic principle. Although it is difficult to estimate gravitational entropy we will attempt to be semi-quantitative in implementing the idea.

Let us consider one halo with mass $M(halo) = 10^{11}M_\odot$ and radius $R_{halo} = 10^{17}r_S(M_\odot) \simeq 10^4 pc$. Applying the holographic principle with regard to the clumping effect would give an overestimate for the halo entropy $S_{halo}(t_0) = S_{halo}^{(HOLO)}(t_0)$ which we propose to estimate with a phenomenological clumping factor by the following ansatz,

$$S_{halo}(t_0) = S_{halo}^{(HOLO)}(t_0) \left(\frac{r_S(halo)}{R(halo)} \right)^p \quad (5)$$

where p is a real parameter. Since $r_S(halo) \leq R(halo)$, Eq.(5) ensures that $S_{halo} \leq S_{halo}^{(HOLO)}(t_0)$ provided that $p \geq 0$. Actually the holographic principle requires that $S_{halo} \leq S_{BH}(M_{halo})$ and since $S_{BH} \propto r_S^2$, this requires that $p \geq 2$ in Eq.(5).

^{#3}Our bounds on entropy are consistent with [4]

The value $p = 2$ provides a more realistic upper limit on the present gravitational entropy of the universe $S_{grav}(t_0)$ than Eq.(4). Using our average values for M_{halo} and R_{halo} and a number 10^{12} of halos this gives

$$S_{grav}(t_0) < 10^{110} \quad (6)$$

which is many orders of magnitude below the holographic limit of Eq.(4). The physical reason is that the clumping to one black hole is very incomplete as there are a trillion disjoint halos. If all the halos coalesced to one black hole, and there is no reason to expect this given the present accelerating expansion of the universe, the entropy would reach the maximum value in Eq.(4) of 10^{123} but at present the upper limit is given by Eq. (6).

Lower Limit on Gravitational Entropy.

It is widely believed that most, if not all, galaxies contain at their core a supermassive black hole with mass in the range $10^5 M_\odot$ to $10^9 M_\odot$ with an average mass of about $10^7 M_\odot$. In our simplified model we assume all have this average mass, so each carries an entropy $S_{BH}(\text{supermassive}) \simeq 10^{91}$. Since there are 10^{12} halos this provides the lower limit on the gravitational entropy [7], [8] of

$$S_{grav}(t_0) \gtrsim 10^{103} \quad (7)$$

which together with Eq.(6) provides a seven order of magnitude window for $S_{grav}(t_0)$.

The lower limit in Eq.(7) from the galactic supermassive black holes may be the largest contributor to the entropy of the present universe but this seems to us highly unlikely because they are so very small, occupying $\sim 10^{-36}$ of the volume. Each supermassive black hole is about the size of our solar system or smaller and it seems counterintuitive that essentially all of the entropy is so concentrated. On the other hand, since gravitational entropy grows with clumping, this could be the case.

As already mentioned, gravitational entropy is associated with the clumping of matter because of the long range unscreened nature of the gravitational force. This is why we propose that the majority of the entropy is associated with the largest clumps of generalized matter: the dark matter halos associated with galaxies and clusters of galaxies.

Most Likely Value of Gravitational Entropy.

In the phenomenological formula for clumping, Eq.(5), the parameter p must satisfy $2 \leq p < \infty$ because for $p = 2$ the halo entropy is as high as it can be, being equal to that of the largest single black hole into which it could collapse, while for $p \rightarrow \infty$, the halo has no gravitational entropy beyond that of the supermassive black hole at its core. Thus, our upper and lower limits

$$10^{110} \geq S_{grav}(t_0) \geq 10^{103} \quad (8)$$

correspond to $p = 2$ and $p \rightarrow \infty$ in Eq.(5) respectively. We may include the supermassive black holes in Eq.(5) by noticing [9] that $S_{grav}(t_0) = 10^{(124-7p)}$ and therefore, from Eq.(8), $2 \leq p \leq 3$.

Actually, the power p in Eq.(5) must depend on the halo radius R_{halo} such that $p(R_{halo}) \rightarrow 2$ as $R_{halo} \rightarrow r_S$, the Schwarzschild radius, when the halo collapses to a black hole. For the present non-collapsed status of the halos, $p > 2$ is necessary since the black hole represents a maximum possible entropy. One might also expect p to be dependent on density and therefore radially dependent, but we assume this is mild enough to allow us to obtain order of magnitude estimates by employing constant p .

We believe the pursuit of better understanding of gravitational entropy in clumps of matter with mass above $M_e = 10^{21}$ kg. (see Eq. (11) below) may provide a very fruitful approach towards a satisfactory theory of quantum gravity. The gravitational entropy we are discussing, if it exists, may well be a quantum mechanical phenomenon.

We can apply the same considerations based on Eq.(5) to gravitation within a single star like the Sun. The Sun has $(r_S/R_\odot) \sim 10^{-5}$ and with $p_a = 3$ we find $S_\odot^{(grav)} \sim 10^{72}$, far above the standard $S_\odot \sim 10^{57}$, suggesting a contribution from stars to the gravitational entropy of about $\sim 10^{95}$.

As the gravitating object we consider becomes smaller the relative importance of gravitational entropy to non-gravitational entropy changes. Let us obtain a rough estimate of the mass M_e at which the two contribution are comparable.

Suppose $M_e = \eta M_\odot \simeq 10^{30}\eta$ kg. and so we wish to determine η . We can estimate η by the fact that the gravitational entropy in Eq.(5) is not linear in η but has a different dependence. Let us take a typical density of the putative object to be $\rho = 5\rho_{H_2O} = 5 \times 10^{12} \text{ kg}/(\text{km})^3$. The radius of a sphere with mass M_e is then $R \simeq 4 \times 10^5 \eta^{1/3}$ km. Thus the gravitational entropy from Eq.(5) is

$$S_{grav} = (10^{77}\eta^2) \left(\frac{3\eta}{4 \times 10^5 \eta^{1/3}} \right) \simeq 10^{72} \eta^{8/3} \quad (9)$$

revealing the η dependence of S_{grav} . On the other hand, the non-gravitational entropy may be estimated by counting baryons to give the usual form linear in η

$$S_{non-grav} \simeq 10^{57} \eta. \quad (10)$$

The two contributions, S_{grav} of Eq.(9) and $S_{non-grav}$ of Eq. (10) become comparable when $\eta^{-5/3} \sim 10^{15}$ or $\eta \sim 10^{-9}$. This equality mass M_e is about

$$M_e \simeq 0.1\% M_\oplus \simeq 10^{21} \text{ kg}. \quad (11)$$

If we consider much smaller masses such as a baseball (~ 1 kg) or a primordial black hole with lifetime comparable to the age of the universe ($\sim 10^{12}$ kg), the gravitational entropy becomes totally negligible.

According to our phenomenological clumping ansatz, Eq.(5), the entropy of solar system objects can be larger than conventionally assumed, the Sun by 10^{15} , the Earth by 10^5 . We possess no derivation of this gravitational entropy component and publish this idea only to prompt more mathematically rigorous arguments to estimate the contribution of gravitational clumping to entropy.

Another reason to suspect a large gravitational entropy outside of black holes comes from considering the gravitational collapse of an object of mass, say, $M = 10M_\odot$ which contains $\sim 10^{58}$ nucleons and hence non-gravitational entropy $S \sim 10^{58}$. Under gravitational collapse, it is conventionally believed that the entropy gradually increases, though *not* by orders of magnitude, as the radius decreases toward the Schwarzschild radius. When the trapped surface of a black hole appears, the entropy jumps to $\sim 10^{79}$, an increase of some twenty orders of magnitude. While not excluded, this is intuitively implausible. On the other hand, with the clumping factor of Eq.(5) and the starting density we have employed of $\rho = 5\rho_{H_2O}$, the starting entropy from Eq.(9) is already $\sim 10^{72+8/3} \sim 5 \times 10^{74}$, and less dramatic entropy increase is needed. In fact, S_{halo} is a smooth function as $R \rightarrow r_S$.

There is a second consideration which provides circumstantial evidence for unsuspected gravitational entropy. If, as in Eq.(7), the cosmological entropy is dominated by the supermassive black holes, it implies that almost all the entropy is confined to a trillion objects each of radius $\sim 10^{-6}$ pc occupying $\sim 10^{-33}$ of the halo volume. Altogether they compose only $\sim 10^{-36}$ of the total volume of the visible universe. Although not excluded by any deep principle, it is disconcerting.

Let us attempt to make a somewhat more quantitative argument out of the idea of how entropy grows with gravitational clumping. At last scattering density perturbations in the dark matter were small, $\frac{\delta\rho}{\rho} \sim 10^{-5}$, but today there are regions where $\frac{\delta\rho}{\rho} \sim 1$ where we expect the gravitational entropy has increased enormously even though the entropy in photons per comoving volume has remained relatively constant.

The non-clumped component of the universe expands adiabatically. How do we get the entropy of a clump? Assume the dark matter is in the form of very light particles. For a clump of size $L_{gal} = 10^4$ pc, the lightest particles that can clump are of mass $m \sim 10^{-20}$ eV. Otherwise their associated Compton wavelength may be too large. Recall the galactic mass is $M_{gal} \sim 10^{12}M_\odot \sim 10^{69}\text{GeV}$. If this is all in dark matter (ignore baryons, etc.), then there can be at most $N \sim \frac{M_{gal}}{m} \sim 10^{98}$ dark matter particles in a halo, or about $N_U \sim 10^{110}$ dark matter particles in the universe that are now clumped.

If the dark matter particles start off at rest (similarly to nonthermal axions) but then start to fall into clumps, we can argue that their degrees of freedom get excited, i.e., as the

particles fall into the potential well they gain kinetic energy. So these gravitational d.o.f.s give approximately zero contribution to the total entropy before density perturbations start to grow, but they now contribute $\sim 10^{110}$. If the masses of the dark matter particles are larger, the contribution to the entropy will be proportionally smaller. The mass $m \sim 10^{-20}$ eV provides an approximate upper bound on the gravitational entropy. The lower bound for the entropy in this particular approach would be very small if the dark matter particles are far heavier such as WIMPs at the TeV scale.

Discussion and Conclusions

Entropy is always a subtle concept, nowhere more so than for gravity. This is why we are bold enough to make such approximate estimates of the present gravitational entropy of the visible universe. Our results are concerned only with orders of magnitude and we hope our upper and lower limits 10^{110} and 10^{103} are credible [10].

These already show that the universe's entropy is dominated by gravity, being at least 13 orders of magnitude above the known entropies, each $\simeq 10^{88}$, for photons and relic neutrinos.

Using the clumping idea and an heuristic clumping factor dependent on a parameter p suggests that the gravitational entropy is dominated not by the well known galactic supermassive black holes which contribute $\simeq 10^{103}$ but by a larger, possibly much larger, contribution from the dark matter halos which can provide not more than (for $p \rightarrow 2$) about 10^{110} which is still many orders of magnitude below the holographic bound $\simeq 10^{123}$. Our estimates for the entropy of the universe have an error of one order of magnitude.

Everything we have said about gravitational entropy due to clumping may be nonsensical but, if it exists, it is reasonable to expect it to be non-classical and an effect of quantum gravity like the holographic bound and the black hole entropy. Since string theory has had some success in those two cases, it could help in deciding whether our speculations are idle. More optimistically, the study of gravitational entropy will itself lead to a better theory of quantum gravity, hopefully the correct one.

If our speculations are correct: the contribution of radiation to the total entropy of the Universe is less than 1 part in 10^{16} ; supermassive black holes at galactic cores contribute more but still less, possibly much less, than a few per cent of the total; we propose that the gravitational entropy contained only in stars is already greater than the entropy of electromagnetic radiation; and the gravitational entropy contained in dark matter halos can be the biggest contributor to the entropy of the universe [11].

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- [9] Taking the log of Eq. (6) and holding everything but p fixed we can write

$$\log_{10} S = a + bp$$
 or equivalently $S = 10^{a+bp}$. Now requiring a and b be chosen to satisfies the two boundary conditions set by Eq. (10) gives the result. Note that $p \rightarrow \infty$ corresponds to no additional gravitational entropy from the dark matter.
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